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RECURRENT AND RICCI RECURRENT FINSLER SPACES ADMITTING PROJECTIVE TRANSFORMATION

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Introduction

Let F_n be an n-dimensional Finsler space equipped with fundamental metric function $F(x,\dot{x})$ satisfying the requisite condition for conditions for being a Finsler metric and $g_{ij}(x,\dot{x})$ be the fundamental metric tensor defined by $F^2(x,\dot{x}) = g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^t$, Let $T_j^i(x,\dot{x})$ be any tensor field, the covariant derivative of such a tensor field in the sense of cartain is given by Rund [2]

(1.1) $T_{j|k}^i = \partial_k T_j^i - \dot{\partial}_k T_j^i \dot{\partial}_k G^h + T_j^m \Gamma_{mk}^{xi} - T_m^i \Gamma_{jk}^{xm}$

where $\Gamma_{jk}^{xi}(x,\dot{x})$ are the cartan connection coefficients satisfying the relation

(1.2) (a) $\Gamma_{jk}^{xi} = \Gamma_{kj}^{xi}$, (b) $\dot{\partial}_h \Gamma_{jk}^{xi} \dot{x}^h = 0$

We are quoting the following commutation formulae which shall be used in the later discussion:

(1.3) $T_{j|hk}^i - T_{j|kh}^i = -\dot{\partial}_r T_j^i K_{mhk}^r \dot{x}^m + T_j^r K_{rhk}^i - T_r^i K_{jkh}^r$

(1.4) $\left(\dot{\partial}_k T_j^i\right)_{|h} - \dot{\partial}_k T_{j|k}^i = \dot{\partial}_r T_j^i C_{hk|m}^i x^m - T_j^r \dot{\partial}_k \Gamma_{rh}^{xi} + T_r^i \dot{\partial}_k \Gamma_{jh}^{xr}$

Where curvature tensor field $K_{jkh}^i(x,\dot{x})$ is given by

$$(1.5)\quad K_{jkh}^r = Q_{(h,k)}\left\{\frac{\partial \Gamma_{jh}^{xi}}{\partial x^k} - \frac{\partial \Gamma_{jh}^{xi}}{\partial \dot{x}^r} G_k^r + \Gamma_{mk}^{xi}\Gamma_{jh}^{xm}\right\}$$

Where $Q_{(h,k)}\{\dots\dots\dots\}$ Stands for the interchange of the indices h and k and subtraction thereafter. The curvature tensor field $K_{jkh}^i(x,\dot{x})$ is a homogenous function of degree zero in its directional arguments. The term $C_{jk}^i(x,\dot{x})$ appearing in (1.4) is any tensor field give by the following equation:

$$(1.6)\quad C_{ijk}(x,\dot{x}) = \frac{1}{2}\frac{\partial g_{ij}(x,\dot{x})}{\partial \dot{x}^k} = \frac{1}{4}\frac{\partial^3 F^2(x,\dot{x})}{\partial \dot{x}^i\,\partial \dot{x}^j \partial \dot{x}^k}$$

We now give the following definition which shall be used in the later discussions.

Definition (1.1)

A Finsler space F_n is said to be affinely connected if

(1.7) $C_{ijk|h} = 0$

Also the equation

(1.8) $\dot{\partial}_r G_{hk}^i = 0 \;\; implies\, C_{ijk|h} = 0$ hence an affinely connected space is also characterized by the equation

(1.9) $\dot{\partial}_r \Gamma_{hk}^{xi} = 0$

Definition (1.2)

A Finsler space F_n is said to be recurrent Finsler space of the first order if the curvature tensor field $K_{jhk}^i(x,\dot{x})$ satisfies the relation:

(1.10) $K_{jkh|m}^i = \lambda_m(x,\dot{x}) K_{jkh}^i$,

Where λ_m is a homogenous function of degree zero in directional arguments. Contracting (1.10) with respect to the indices i and h, we get, (1.11) $K_{jk|m} = \lambda_m K_{jk}$

Where

(1.12) $K_{jk} = K^i_{jki}$ is the Ricci tensor.

In such a case the recurrent Finsler space is terred as Ricci-recurrent Finsler space.

We have said in the foregoing lines that the curvature tensor $K^i_{jkh}(x,\dot{x})$homogeneous of degree zero in its directional arguments therefore$K^i_{jk}(x,\dot{x})$ is also homogenous of degree zero in its directional arguments and hence we shall have-

(1.13) $\dot{\partial}_r K_{jk}\dot{x}^r = 0$

We now consider an infinitesimal point transformation given by

(1.14) $\bar{x}^i = x^i + v^i_{(x)}dt,$

where$v^i_{(x)}$ stands for a covariant vector field defined over the domain of the space under consideration and dt is an infinitesimal constant. The Fnisler space F_n is said to admit an infinitesimal projective transformation with respect to the vector field $v^i_{(x)}$ if the Lie-derivative of the connection coefficients $\Gamma^{*i}_{jk}$with respect to (1.14) has the form

(1.15) $\mathcal{L}_v \Gamma^{*i}_{hk} = -\delta^i_j p_k - \delta^i_k p_j$

Where $\mathcal{L}_v$ denotes the operation of Lie-differentiation and $p(x,\dot{x})$ is an arbitrary scalar function positively homogeneous of degree one in x^i and satisfies-

(1.16) (a) $p_k = \dot{\partial}_k p,$ (b) $p_{jk} = \dot{\partial}_k \dot{\partial}_j\, p,$

Because of homogeneity property of $p(x,\dot{x})$we have

(1.17) (a) $p_k \dot{x}^k = p,$ (b) $p_{jk}\dot{x}^k = 0$

It has been given in [2] that

(1.18) $\dot{\partial}_k\left(\mathcal{L}_v T^i_j\right) - \mathcal{L}_v\left(\dot{\partial}_k T^i_j\right) = 0$

(1.19) $\left(\mathcal{L}_v T^i_j\right)_{|k} - \mathcal{L}_v T^i_{j|k} = -T^i_j \mathcal{L}_v \Gamma^{*i}_{hk} + T^i_h \Gamma^{*h}_{jk} + \dot{\partial}_k T^i_j \mathcal{L}_v \Gamma^{*h}_{rk}\dot{x}^r.$

(1.20) $\left(\mathcal{L}_v \Gamma^{*i}_{jk}\right)_{|h} - \left(\mathcal{L}_v \Gamma^{*i}_{jh}\right)_{|k} = \mathcal{L}_v K^i_{jkh} + 2\dot{\partial}_r \Gamma^{*h}_{j[k}\mathcal{L}_v \Gamma^{*r}_{h]s}\dot{x}^s,$

Recurrent and Ricci Recurrent Finsler Spaces Admitting Projective Transformation

With the help of (1.15) and (1.20), we can easily get

(2.1) $\mathcal{L}_v K^i_{jkh} = 2\left[\delta^i_j p_{[h/k]} + \delta^i_{[h\,p_j]}{}_h + \dot{\partial}_{[h\,\Gamma^{*i}_{k]j}\,p]}\right]$

If at this stage we assume that the space under consideration is affinity then in view of (1.9) the equation (2.1) assumes the following form-

(2.2) $\mathcal{L}_v K^i_{jkh} = 2\left\{\delta^i_j p_{[h/k]} + \delta^i_{[h\,p_{j/k]}}\right\}$

we now contract (2.2) with respect to the indices i and h and thereafter use the relation (1.12) and get

(2.3) $\mathcal{L}_v K_{jk} = np_{j|k} - p_{k|j} = \phi_{jk}(say)$

We at this stage multiply (2.3) by $\dot{x}^j\dot{x}^k$ and use the homogeneity property of $p(x,\dot{x})$ and get

(2.4) $\mathcal{L}_v K_{jk}\dot{x}^j\dot{x}^k = (n-1)p_{|i}\dot{x}^i$

Multiplying (2.2) by $\dot{x}^j$ and then simplifying, we get

(2.5) $\mathcal{L}_v K^i_{jkh}\dot{x}^j = 2\left\{\dot{x}^i p_{[h/k]} + \delta^i_{[h\,p_{j/k]}}\right\}$

With the help of (2.3), we can easily deduce the following relations

(2.6) $\phi_{jk}\dot{x}^j = n\; p|k - p_{k|j}\dot{x}^j,$

(2.7) $\phi_{jk}\dot{x}^k = n\,p_{j|k}\dot{x}^k - p|j,$

(2.8) $\phi_{jk}\dot{x}^j\dot{x}^k = (n-1)p|j\dot{x}^j,$

(2.9) $\dot{\partial}_r\phi_{jk}\dot{x}^j\dot{x}^k = 0$

Differentiating (1.11) covariantly with respect to x^m, in the sense of cartan and then using (1.11) itself, we get

(2.10) $K_{ij|mh} = (\lambda_{m|h} + \lambda_m\lambda_h)K_{ij}$

Commutating (2.10) with respect to the indices h and m, we get

(2.11) $K_{ij|hm} - K_{ij|mh} = (\lambda_{h|m} - \lambda_{m|h})K_{ij}$

Using the commutation formula (1.3) in (2.11), we get

(2.12) $\dot{\partial}_r K_{ij} K^r_{phm}\dot{x}^p + K_{ir}K^r_{jhm} + K_{rj}K^r_{ihm} = \dot{\partial}_r\lambda\, K^r_{phm}\dot{x}^p K_{ij}$

Simplifying (2.12) after multiplying it with $\dot{x}^i\dot{x}^j$, we get

(2.13) $K^r_{phm}\dot{x}^p\{(K_{ij}\dot{x}^i\dot{x}^j)(\dot{\partial}_r\lambda) - \dot{\partial}_r(K_{ij}\dot{x}^i\dot{x}^j)\} = 0$

With the help of (2.13) we arrive at either of the following two conclusions

(2.14) (a) $K^r_{phm}\dot{x}^p = 0$ or

(b) $(K_{ij}\dot{x}^i\dot{x}^j)(\dot{\partial}_r\lambda) - \dot{\partial}_r(K_{ij}\dot{x}^i\dot{x}^j) = 0$

Therefore, we can state:

Theorem (2.1)

In a Ricci-recurrent Finsler space either of the following two conclusions always hold,

Either $K^r_{phm}\dot{x}^p = 0$ or $(K_{ij}\dot{x}^i\dot{x}^j)(\dot{\partial}_r\lambda) - \dot{\partial}_r(K_{ij}\dot{x}^i\dot{x}^j) = 0$

We now consider the Lie-derivative of (2.12) and thereafter use the equations (1.15), (1.16), (1.18), (2.3), (2.5), (2.6), (2.7), (2.8) and (2.9), we get

(2.15) $K^r_{phm}\dot{x}^p\dot{\partial}_r\phi_{ij} + (p_{|h}\dot{\partial}_m K_{ij} - p_{|m}\dot{\partial}_h K_{ij}) + \phi_{ir}K^r_{jhm} + 2K_{ij}(p_{m|h} - p_{h|m}) + K_{im}p_{j|h}$

$-K_{ih}p_{j|m} + \phi_{rj}K^r_{ihm} + K_{mj}p_{i|h} - pK_{hj}p_{i|m}$

$= \mathcal{L}_v(\dot{\partial}_r\lambda)K^r_{phm}\dot{x}^p K_{ij} + K^r_{phm}\dot{x}^p\phi_{ij}\dot{\partial}_r\lambda + K_{ij}(p_{|h}\dot{\partial}_m\lambda - p_{|m}\dot{\partial}_h\lambda)$

Multiplying (2.15) by $\dot{x}^i\dot{x}^j$ and there using (1.16), we get

(2.16) $p_{|h}\dot{\partial}_m(K_{ij}\dot{x}^i\dot{x}^j) - p_{|m}\dot{\partial}_h(K_{ij}\dot{x}^i\dot{x}^j) + K^r_{ihm}(n-1)p_{|r} + (n-1)K^r_{ihm}\dot{x}^i p_{r|j}\dot{x}^j +$

$2K_{ij}\dot{x}^i\dot{x}^j(p_{m|h} - p_{h|m}) = \mathcal{L}_v(\dot{\partial}_r\lambda)K^r_{phm}\dot{x}^p K_{ij}\dot{x}^i\dot{x}^j + (n-1)K^r_{phm}\dot{x}^p p_{|i}\dot{x}^i$

$+K_{ij}\dot{x}^i\dot{x}^j(p_{|h}\dot{\partial}_m\lambda - p_{|m}\dot{\partial}_h\lambda)$

We now take the Lie-derivative of (2.14b) and thereafter use the equation (2.3), (2.7), (2.8) and (2.9) and get

(2.17) $(K_{ij}\dot{x}^i\dot{x}^j)\mathcal{L}_v(\dot{\partial}_r\lambda) + (n-1)p_{|k}\dot{x}^k\dot{\partial}_r\lambda = (n-1)p_{r|k}\dot{x}^k + (n-1)p_{|r}$

Using (2.14b) and (2.17) in (2.16), we get

(2.18) &$(K_{ij})\dot{x}^i\dot{x}^j p_{[m|h]} = 0$

An obvious consequence of (2.18) is

(2.19) (a) $K_{ij}\dot{x}^i\dot{x}^j = 0$ or (b) $p_{[m|h]} = 0$

Therefore, we can state the following:

Theorem (2.2)

In an affinity connected Ricci-recurrent Finsler space admitting infinitesimal projective transformation either of the following two always hold

either $K_{ij}\dot{x}^i\dot{x}^j = 0$ or $p_{[h|m]} = p_{[m|h]}$

We know that recurrent Finsler space become Ricci-recurrent hence with the help of theorems (2.1) and (2.2), we can state the following:

Theorem (2.3)
A recurrent Finsler space always satisfies either of the following two relations:
Either $K^r_{phm}\dot{x}^p = 0$ or $(K_{ij}\dot{x}^i\dot{x}^j)(\dot{\partial}_r\lambda) - \dot{\partial}_r(K_{ij}\dot{x}^i\dot{x}^j) = 0$

Theorem (2.4)
In an affinity connected recurrent Finsler space admitting infinitesimal projective transformation either of the following two always holds:
either $K_{ij}\dot{x}^i\dot{x}^j = 0$ or $p_{[m|h]} = p_{[h|m]}$
We now consider the Lie-derivative of (1.11) and get
(2.20) $\mathcal{L}_v K_{ij|h} = (\mathcal{L}_v\lambda_h)K_{ij} + \lambda_h K_{ij}$
Applying the commutation formula (1.19) for the Ricci-tensor $K_{ij}(x,\dot{x})$ we get,
(2.21) $\left(\mathcal{L}_v K_{ij}\right)_{|h} - \mathcal{L}_v K_{ij|h} = \dot{\partial}_k K_{ij}\mathcal{L}_v\Gamma^{\dot{x}k}_{rh}\dot{x}^r + K_{ij}\mathcal{L}_v\Gamma^{\dot{x}l}_{ih} + K_{il}\mathcal{L}_v\Gamma^{\dot{x}l}_{jh}$
Using (1.15), (1.16) and (2.3) in (2.20), we get
(2.22) $\phi_{ij|h} + p\dot{\partial}_h K_{ij} + 2K_{ij}p_h + K_{hj}p_i + K_{ih}p_j = K_{ij}\mathcal{L}_v\lambda_h + \lambda_h\phi_{ij}$
Multiplying (2.22) by $\dot{x}^i\dot{x}^j$ and thereafter using (1.16) and (2.8), we get
(2.23) $(n-1)(p_{|kh} - \lambda_h p_{|k})\dot{x}^k + p\,\dot{\partial}_h(K_{ij}\dot{x}^i\dot{x}^j) + K_{ij}\dot{x}^i\dot{x}^j(2P_h - \mathcal{L}_v\lambda_h) = 0$
Multiplying (2.23) by $\dot{x}^h$ and then making use of (1.13), we get
(2.24) $(n-1)(p_{|kh} - \lambda_h p_{|k})\dot{x}^k\dot{x}^h + K_{ij}\dot{x}^i\dot{x}^j\{4P - \mathcal{L}_v\lambda_h\dot{x}^h\} = 0$
Therefore, we can state the following:

Theorem (2.5)
In a Ricci-recurrent affinity connected Finsler space admitting infinitesimal projective transformation the relation (2.24) always holds.
Using (2.19a) in (2.24), we get
(2.25) $(p_{|kh} - \lambda_h p_{|k})\dot{x}^k\dot{x}^h = 0$
Therefore, we can state:

Theorem (2.6)
In a Ricci-recurrent affinity connected Finsler space admitting infinitesimal projective transformation satisfies $K_{ij}\dot{x}^i\dot{x}^j$, Then (2.25) always holds.
We know that every recurrent Finsler space is Ricci-recurrent, Therefore, we can state:

Theorem (2.7)
In a recurrent affinity connected Finsler space admitting infinitesimal projective transformation the relation (2.24) necessarily holds.

Theorem (2.8)
In a recurrent affinity connected Finsler space admitting infinitesimal projective transformation satisfies $K_{ij}\dot{x}^i\dot{x}^j = 0$ then $\dot{x}^k\dot{x}^h(p_{|kh} - \lambda_h p_{|k}) = 0$ always holds.

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