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ON AFFINE, CONFORMAL AND SPECIAL PROJECTIVE MOTION IN R^+-Fn*

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Introduction

In one of his papers Vranceanu, G. [8] has defined a non – symmetric Connection $\Gamma^i_{jk}\left(\neq \Gamma^i_{kj}\right)$ in an n – dimensional An, we extend this concept to the theory of n – dimensional Finsler space.

Let us consider an n – dimensional Finsler space $F_n$with non – symmetric connection $\Gamma^i_{jk}\left(x,\dot{x}\right)\left(\neq \Gamma^i_{kj}\left(x,\dot{x}\right)\right.$ which is based on non – symmetric fundamental tensor $g_{ij}\left(x,\dot{x}\right)\left(\neq g_{ji}\left(x,\dot{x}\right)\right)$, such a Finsler space equipped with non – symmetric connection Γ^i_{jk} will be denoted by F_n^* to distinguish it from F_n that is equipped with symmetric connection coefficient based on symmetric metric tensor $g_{ij}\left(x,\dot{x}\right) = g_{ji}\left(x,\dot{x}\right)$

Let us write

$$\textbf{(1.1)} \qquad \Gamma^i_{jk} = M^i_{jk} + \frac{1}{2} N^i_{jk}$$

Where M^i_{jk} and $\frac{1}{2} N^i_{jk}$ are the symmetric and skew – symmetric parts of $\Gamma^i_{jk}\left(x,\dot{x}\right)$ defined as below

$$\textbf{(1.2)} \qquad \tilde{\Gamma}^i_{jk}\left(x,\dot{x}\right) = M^i_{jk} - \frac{1}{2} N^i_{jk}$$

With the help of (1.1) and (1.2), we get

$$\textbf{(1.3)} \qquad \tilde{\Gamma}^i_{jk}\left(x,\dot{x}\right) = \tilde{\Gamma}^i_{kj}\left(x,\dot{x}\right)$$

Following E. Cartan [1], let a vertical stroke (1), followed by an index denote covariant derivative with respect to x. Here we define covariant derivative of any contra variant vector field $X^i\left(x,\dot{x}\right)$ in two distinct ways [8] as follows :

$$\textbf{(1.4)} \qquad X^i {}^{+}|_j = \partial_j X^i - \left(\dot{\partial}_m X^i\right)\Gamma^m_{kj}\dot{x}^k + X^k\Gamma^i_{kj}$$

and

$$\textbf{(1.5)} \qquad X^i {}^{-}|_j = \partial_j X^i - \left(\dot{\partial}_m X^i\right)\tilde{\Gamma}^m_{kj}\dot{x}^k + X^k\tilde{\Gamma}^i_{kj}$$

Where a positive sign (+) below an index and followed by a vertical stroke indicates that the covariant derivative has been formed with respect to the connection Γ^i_{jk}, as far as that index is concerned. Similarly, a negative sign below an index and followed by vertical stroke indicates that the covirant derivative has been formed with respect to connection $\tilde{\Gamma}^i_{jk}$ concerning that index. The covariant differentiation defined in (1.4) and (1.5) will be called $\oplus$ - covariant differentiation of $X^i\left(x,\dot{x}\right)$ with respect to x^j and Θ -covariant differentiation of with respect to x^j respectively.

Differentiating (1.4), $\oplus$ - covariantly with respect to x^k and talking the skew symmetric part of the result they obtained with respect to j and k, we obtain the following commutation formula.

$$\textbf{(1.6)} \qquad X^i {}^{+}|_{jk} - X^i {}^{+}|_{kj} = -\left(\dot{\partial}_m X^i\right)R^m_{pjk}\dot{x}^p + X^m R^i_{mjk} + X^i {}^{+}|_m \; N^m_{kj},$$

Where

$$\textbf{(1.7)} \qquad R^h_{ijk} \overset{def.}{=} \partial_k\Gamma^h_{ij} - \partial_j\Gamma^h_{ik} + \dot{\partial}_m\Gamma^h_{ij}\Gamma^m_{sk}\dot{x}^s - \dot{\partial}_m\Gamma^h_{ik}\Gamma^m_{sj}\dot{x}^s +$$
$$+\Gamma^p_{ij}\Gamma^h_{pk} - \Gamma^p_{ik}\Gamma^h_{pj}.$$

Similarly, differentiating (1.5), Θ -covariantly with respect to x^k and proceeding as above, we get

$$\textbf{(1.8)} \qquad X^i {}^{-}|_{jk} - X^i {}^{-}|_{kj} = -\left(\dot{\partial}_m X^i\right)\tilde{R}^m_{pjk}\dot{x}^p + X^m\tilde{R}^i_{mjk} + X^i {}^{-}|_m \; N^m_{jk},$$

Where

(1.9) $$\tilde{R}^h_{ijk} \overset{def.}{=} \partial_k \tilde{\Gamma}^h_{ij} - \partial_j \tilde{\Gamma}^h_{ik} + \dot{\partial}_m \tilde{\Gamma}^h_{ik} \tilde{\Gamma}^m_{sj} \dot{x}^s - \dot{\partial}_m \tilde{\Gamma}^h_{ij} \tilde{\Gamma}^m_{sk} \dot{x}^s + + \tilde{\Gamma}^p_{ij} \tilde{\Gamma}^h_{pk} - \tilde{\Gamma}^p_{ik} \tilde{\Gamma}^h_{pj}.$$

The entities R^h_{ijk} and $\tilde{R}^h_{ijk}$ defined in (1.7) and (1.9) are called “Curvature tensors” which arise due to the duality in the nature of covariant derivatives defined in (1.4) and (1.5), we will extensively use the following identities and notations in the sequel

(1.10) **(a)** $\dot{x}^i\ {}^+|_k = 0 = \dot{x}^i\ {}^-|_k,$

(b) $R^i_{jk} = R^i_{hjk} \dot{x}^h,$

(c) $R^i_j = R^i_{hj} \dot{x}^h,$

(d) $R^i_{hjk} = -R^i_{hkj}, R^i_{jk} = -R^i_{kj},$

(e) $N^i_{jk} = -N^i_{kj} = \Gamma^i_{jk} - \Gamma^i_{kj},$

(f) $\Gamma^i_{hjk} = \dot{\partial}_h \Gamma^i_{jk}$

Existence of Affine Motion in AN R+-Recurrent Space

We consider the existence of R^+-recurrent space, for this purpose we consider an infinitesimal transformation

(2.1) $\overline{x}^i = \dot{x}^i + \mathrm{v}^i(x)dt,$

Where $\mathrm{v}^i(x)$ an arbitrary contravariant is vector field and dt is an infinitesimal point constant. This point transformation when considered, at each point in R^+-recurrent space, is called affine motion when and only when

(2.2) $\pounds_{\mathrm{v}} \Gamma^i_{jk} = 0,$

Where $\pounds_{\mathrm{v}}$ denotes the operator of Lie-differentiation and Γ^i_{jk} is the non-symmetric connection coefficient. By virtue of the above point transformation, the Lie-derivative of any arbitrary tensor field $T^i_j(x, \dot{x})$ and the connection coefficient $\Gamma^i_{jk}(x, \dot{x})\left(\neq \Gamma^i_{kj}(x, \dot{x})\right)$ in view of the $\oplus$ - covariant derivative are given by Gupta [2] as

(2.3) $$\pounds_{\mathrm{v}} T^i_j(x, \dot{x}) = T^i_j\ {}^+|_k\ \mathrm{v}^k + \left(\dot{\partial}_k T^i_j\right)\left(\mathrm{v}^k\ {}^+|_h\right)\dot{x}^h - -T^k_j \mathrm{v}^i\ {}^+|_k + T^i_k \mathrm{v}^k\ {}^+|_j,$$

And

(2.4) $$\pounds_{\mathrm{v}} \Gamma^i_{jk}(x, \dot{x}) = \left(\mathrm{v}^i\ {}^+|_j\right) {}^+|_k + \left(\dot{\partial}_r \Gamma^i_{jk}\right) \mathrm{v}^r\ {}^+|_h\ \dot{x}^h + \mathrm{v}^h R^i_{jkh},$$

Where R^i_{jkh} has been given by (1.7).

Between the operators $\pounds_{\mathrm{v}}, \dot{\partial}$ and ${}^+|_k$, we have the following commutation formulae Gupta [2]

(2.5) $\pounds_{\mathrm{v}}\left(\dot{\partial}_k T^i_j\right) - \dot{\partial}_k\left(\pounds_{\mathrm{v}} T^i_j\right) = 0,$

(2.6) $$\pounds_{\mathrm{v}}\left(T^i_j\ {}^+|_k\right) - \left(\pounds_{\mathrm{v}} T^i_j\right) {}^+|_k = T^h_j \pounds_{\mathrm{v}} \Gamma^i_{hk} - T^i_h \pounds_{\mathrm{v}} \Gamma^h_{jk} - -\left(\dot{\partial}_h T^i_j\right)\left(\pounds_{\mathrm{v}} \Gamma^h_{sk}\right)\dot{x}^s,$$

And

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(2.7) $\left(\pounds_{v}\Gamma^{i}_{hj}\right)^{+}|_{k} - \left(\pounds_{v}\Gamma^{i}_{hk}\right)^{+}|_{j} = \pounds_{v}R^{i}_{hjk} + \dot{x}^{\ell}\Gamma^{i}_{rhj}\Gamma^{r}_{\ell k} -$

$-\dot{x}^{\ell}\Gamma^{i}_{rhk}\pounds_{v}\Gamma^{r}_{\ell j} + N^{r}_{kj}\pounds_{v}\Gamma^{i}_{hr}$,

Where R^{i}_{hjk} , N^{i}_{jk} has been Γ^{i}_{rhk} have been given by (1.1), (1.7) and (1.9f) respectively. We now give the following definitions which we shall use in the later discussion.

Definition (2.1)

A Finsler space F^{*}_{n} equipped with non-symmetric connection $\Gamma^{i}_{jk}\left(\neq \Gamma^{i}_{kj}\right)$ is said to be a R^{+}-recurrent F^{*}_{n} if

(2.8) $R^{i}_{jkh}{}^{+}|_{\ell} = \mathrm{a}_{\ell}R^{i}_{jkh}, \quad R^{i}_{jkh} \neq 0$

Where $\mathrm{a}_{\ell}(x)$ is a non-null recurrence vector.

DEFINITION (2.2):

An n-dimensional F^{*}_{n} equipped with non-symmetric connection is said to be an affinely connected R^{+}-space if

(2.9) $\dot{\partial}_{\ell}\Gamma^{i}_{jk} = 0$ holds.

Using the equations (2.7) and (2.2), we get the Lie-derivative of $R^{i}_{jkh}(x,\dot{x})$ as

(2.10) $\pounds_{v}R^{i}_{jhk} = 0$

Taking the Lie-derivative of both sides of (2.8) and thereafter using the equations (2.2), (2.6) and (2.10), we get

(2.11) $\pounds_{v}\mathrm{a}_{\ell}R^{i}_{jkh} = 0$

Consequently, if the R^{+}-recurrent F^{*}_{n} is considered to be non-flat, (a non flat F^{*}_{n} is characterized by $R^{i}_{jkh} \neq 0$) then we have

(2.12) $\pounds_{v}\mathrm{a}_{\ell} = 0$

Thus, with the help of (2.12) we can say that the recurrence vector of a non-flat R^{+}-recurrent F^{*}_{n} must be a Lie-invariant. Next, we propose to study a R^{+}-recurrent F^{*}_{n} admitting an infinitesimal point transformation (2.1) satisfying (2.12). For brevity we shall now onwards call such a restricted space as R^{+}-recurrent space.

The Vanishing of $\pounds_{v}R^{i}_{jkh}(x,\dot{x})$ **:** In this section first of all we prove the following Lemma.

Lemma (3.1)

In a R^{+}-recurrent F^{*}_{n}, if the recurrence vector $\mathrm{a}_{s}(x)$ is a gradient one, we have $\mathrm{a}_{s}v^{s} =$ constant.

Proof:

For brevity, we write $\sim = \mathrm{a}_{m}v^{m}$, with the help of equations (2.3) and (2.12), we get

(3.1) $\pounds_{v}\mathrm{a}_{m} = \mathrm{a}_{m}{}^{+}|_{s}\, v^{s} + \mathrm{a}_{s}v^{s}{}^{+}|_{m}$.

If we now assume that $\mathrm{a}_{m}{}^{+}|_{s} = \mathrm{a}_{s}{}^{+}|_{m}$ then it is obvious that $\sim{}^{+}|_{s} = 0$, which will mean that $\sim = \mathrm{a}_{m}v^{m}$ is a constant.

By virtue of (2.3), the Lie-derivative of $R^{i}_{jkh}(x,\dot{x})$ in view of (2.8) can be written as

(3.2) $\pounds_{v}R^{i}_{jkh} = \sim R^{i}_{jkh} - R^{p}_{jkh}v^{i}{}^{+}|_{p} + R^{i}_{pkh}v^{p}{}^{+}|_{j} + R^{i}_{jph}v^{p}{}^{+}|_{k} +$

$$+R^{i}_{jkp}\mathrm{v}^{p}\ {}^{+}|_{h} + \left(\dot{\partial}_{p}R^{i}_{jkh}\right)\mathrm{v}^{p}\ {}^{+}|_{m}\ \dot{x}^{m}.$$

We use the commutation formula (1.6) for $R^{i}_{jkh}\left(x,\dot{x}\right)$, and get

(3.3) $$R^{i}_{jkh}\ {}^{+}|_{\ell m}\ R^{i}_{jkh}\ {}^{+}|_{m\ell} = -\left(\dot{\partial}_{p}R^{i}_{jkh}\right)R^{p}_{q\ell m}\dot{x}^{q} + R^{p}_{jkh}R^{i}_{p\ell m} -$$
$$-R^{i}_{pkh}R^{p}_{j\ell m} - R^{i}_{jph}R^{p}_{k\ell m} - R^{i}_{jkp}R^{p}_{h\ell m} +$$
$$+\mathrm{a}_{p}R^{i}_{jkh}N^{p}_{m\ell}.$$

(3.3) in view of the definition (2.8) gives

(3.4) $$\left(\mathrm{a}_{\ell}\ {}^{+}|_{m} - \mathrm{a}_{m}\ {}^{+}|_{\ell}\right)R^{i}_{jkh} = -\left(\dot{\partial}_{p}R^{i}_{jkh}\right)R^{p}_{q\ell m}\dot{x}^{q} + R^{p}_{jkh}R^{i}_{p\ell m} -$$
$$-R^{i}_{pkh}R^{p}_{j\ell m} - R^{i}_{jph}R^{p}_{k\ell m} - R^{i}_{jkp}R^{p}_{h\ell m} +$$
$$+\mathrm{a}_{p}R^{i}_{jkh}N^{p}_{m\ell}.$$

Now, we assume that ~ is not a constant, then from Lemma (3.1), we can find the following

(3.5) $${}_{\ell m}\left(x\right) \overset{def.}{=} \left(\mathrm{a}_{\ell}\ {}^{+}|_{m} - \mathrm{a}_{m}\ {}^{+}|_{\ell}\right) \neq 0$$

Now, we consider a suitable non-symmetric tensor Ψ^{jk} which satisfies that relation

(3.6) $$R^{i}_{jkh}\Psi^{kh} = \mathrm{v}^{i}\ {}^{+}|_{j}$$

We now multiply (3.4) by $\Psi^{\ell m}$ and sum it over ℓ and m and get

(3.7) $${}_{\ell m}\Psi^{\ell m}R^{i}_{jkh} = -\left(\dot{\partial}_{p}R^{i}_{jkh}\right)\mathrm{v}^{p}\ {}^{+}|_{q}\ \dot{x}^{q} + R^{p}_{jkh}\mathrm{v}^{i}\ {}^{+}|_{p} -$$
$$-R^{i}_{pkh}\mathrm{v}^{p}\ {}^{+}|_{j} - R^{i}_{jph}\mathrm{v}^{p}\ {}^{+}|_{k} - R^{i}_{jkp}\mathrm{v}^{p}\ {}^{+}|_{h} + dR^{i}_{jkh}$$

Where

(3.8) $$d = \mathrm{a}_{p}N^{p}_{m\ell}\Psi^{m\ell}$$

Comparing (3.7) with (3.2), we get

(3.9) $$\pounds_{\mathrm{v}}R^{i}_{jkh} = \left(\sim - {}_{\ell m}\Psi^{\ell m} + d\right)R^{i}_{jkh}.$$

Form (3.9) it is almost clear that $\pounds_{\mathrm{v}}R^{i}_{jkh}$ vanishes if and only if $\sim = {}_{\ell m}\Psi^{\ell m} - d$.

With the help of (3.2) and (3.4), we can construct the following identity

(3.10) $${}_{\ell m}\pounds_{\mathrm{v}}R^{i}_{jkh} = R^{p}_{jkh}\left(\sim R^{i}_{p\ell m} - {}_{\ell m}\mathrm{v}^{i}\ {}^{+}|_{p}\right) -$$
$$-R^{i}_{pkh}\left(\sim R^{p}_{j\ell m} - {}_{\ell m}\mathrm{v}^{p}\ {}^{+}|_{j}\right) -$$
$$-R^{i}_{jph}\left(\sim R^{p}_{k\ell m} - {}_{\ell m}\mathrm{v}^{p}\ {}^{+}|_{k}\right) -$$
$$-R^{i}_{jkp}\left(\sim R^{p}_{h\ell m} - {}_{\ell m}\mathrm{v}^{p}\ {}^{+}|_{h}\right) -$$
$$-\left(\dot{\partial}_{p}R^{i}_{jkh}\right)\left(\sim R^{p}_{q\ell m} - {}_{\ell m}\mathrm{v}^{p}\ {}^{+}|_{q}\right)\dot{x}^{q} + \sim\mathrm{a}_{p}R^{i}_{jkh}N^{p}_{m\ell}$$

Form (3.10) we conclude that if the skew -symmetric part $N^{p}_{m\ell}$ of the non-symmetric connection Γ^{i}_{jk} vanishes then $\pounds_{\mathrm{v}}R^{i}_{jkh} = 0$ if

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(3.11) $\sim R^{i}_{jkh} = \ _{kh}\mathrm{v}^{i\ +}|_{j}$.

Whereas, if the skew part of the non -symmetric connection does not vanish then $£_{\mathrm{v}} R^{i}_{jkh} = 0$ if

(3.12) **(a)** $\sim R^{i}_{jkh} = \ _{kh}\mathrm{v}^{i\ +}|_{j}$

And

(b) $\sim \mathrm{a}_{p} R^{i}_{jkh} N^{p}_{m\ell} = 0$,

Where v^{i} does not mean a parallel vector.

We now give the following definition:

Definition (3.1)

A R^{+}-recurrent F^{*}_{n} satisfying $\mathrm{a}_{m}\mathrm{v}^{m} \neq$ constant is called a special one of the first kind.

Now, we again come back to the case $\mathrm{a}_{m}\mathrm{v}^{m} =$constant of the foregoing Lemma (3.1)then (3.4) is replaced by

$$\textbf{(3.13)}\quad -\left(\dot{\partial}_{p} R^{i}_{jkh}\right) R^{p}_{q\ell m} \dot{x}^{q} + R^{p}_{jkh} R^{i}_{p\ell m} - R^{i}_{pkh} R^{p}_{j\ell m} - R^{i}_{jph} R^{p}_{k\ell m} - -R^{i}_{jkp} R^{p}_{h\ell m} + \mathrm{a}_{p} R^{i}_{jkh} N^{p}_{m\ell} = 0 .$$

In view of the equation (3.6), multiplying (3.13) by $\Psi^{\ell m}$ and summing over ℓ and m, we get

$$\textbf{(3.14)}\quad -\left(\dot{\partial}_{p} R^{i}_{jkh}\right) \mathrm{v}^{p\ +}|_{q}\ \dot{x}^{q} + R^{p}_{jkh} \mathrm{v}^{i\ +}|_{p} - R^{i}_{pkh} \mathrm{v}^{p\ +}|_{j} - R^{i}_{jph} \mathrm{v}^{p\ +}|_{k} - -R^{i}_{jkp} \mathrm{v}^{p\ +}|_{h} + d R^{i}_{jkh} = 0 ,$$

Where d has been given by (3.8).
We now introduce (3.14) into the right hand side of (3.2) and get

$$\textbf{(3.15)}\quad £_{\mathrm{v}} R^{i}_{jkh} = \left(\sim + d\right) R^{i}_{jkh} ,$$

Where $\sim = \mathrm{a}_{m}\mathrm{v}^{m}$ and d has been given by (3.8).

With the help of (3.15), we conclude that if $\sim + d = 0$ then $£_{\mathrm{v}} R^{i}_{jkh} = 0$ holds. we now give the following definition :

Definition (3.2)

When $\sim = \mathrm{a}_{m}\mathrm{v}^{m} =$constant holds good, a R^{+}-recurrent F^{*}_{n} is said to be a special one F^{*}_{n} of the second kind.

We now summaries all these results in the form of following theorems.

Theorem (3.1)

In a special recurrent F^{*}_{n} of the first kind, if F^{*}_{n} satisfies (3.12a) and (3.12b) both then $£_{\mathrm{v}} R^{i}_{jkh} = 0$ holds good.

Theorem (3.2)

In a special R^{+}-recurrent F^{*}_{n} of the first kind, if F^{*}_{n} satisfies (3.11) then $£_{\mathrm{v}} R^{i}_{jkh} = 0$ holds good only when the skew - symmetric part $N^{p}_{m\ell}$ of the non-symmetric connection $\Gamma^{p}_{m\ell}$ vanishes.

Theorem (3.3)

In a special R^{+}-recurrent F^{*}_{n} of the second kind, if $\sim + d = 0$ then $£_{\mathrm{v}} R^{i}_{jkh} = 0$ hold identically where $\sim = \mathrm{a}_{m}\mathrm{v}^{m}$ and d has been given by (3.8).

Theorem (3.4)

In a symmetric R^{+}-recurrent F_n^{*} of the second kind $\pounds_{v} R^{i}_{jkh} = 0$ holds if $d = 0$.

Complete Condition

We shall here find a necessary and sufficient condition under which (3.11) holds. In view of the assumption (2.12), we have

(4.1) $\pounds_{v}\, a_{m} = a_{m}\,{}^{+}|_{s}\, v^{s} + \left(a_{s} v^{s}\right){}^{+}|_{m} - a_{s}\,{}^{+}|_{m}\, v^{s} = 0.$

(4.1) by virtue of (3.5) reduces to

(4.2) $\sim{}^{+}|_{m} + {}_{ms} v^{s} = 0.$

In view of (2.3), the Lie-derivative of ${}_{\ell m}(x)$ is given by

(4.3) $\pounds_{v}\, {}_{\ell m}(x) = {}_{\ell m}\,{}^{+}|_{p}\, v^{p} + {}_{pm} v^{p}\,{}^{+}|_{\ell} + {}_{\ell p} v^{p}\,{}^{+}|_{m}.$

(4.3) by virtue of (2.6), (2.12) and (3.5) reduce into the following form

(4.4) $\pounds_{v}\, {}_{\ell m} = 0.$

Differentiating (3.4) $\oplus$-covariantly with respect to x^{n} and using (2.8) and (3.4), we get

(4.5) $${}_{\ell m}\,{}^{+}|_{n}\, R^{i}_{jkh} = a_{n}\Big[R^{p}_{jkh} R^{i}_{p\ell m} - R^{i}_{pkh} R^{p}_{j\ell m} - R^{i}_{jph} R^{p}_{k\ell m} - R^{i}_{jkp} R^{p}_{h\ell m} - \\ -\left(\dot{\partial}_{p} R^{i}_{jkh}\right) R^{p}_{q\ell m}\dot{x}^{q}\Big] + a_{p}\,{}^{+}|_{n}\, R^{i}_{jkh} N^{p}_{m\ell} + \\ + a_{p} R^{i}_{jkh} N^{p}_{m\ell}\,{}^{+}|_{n}.$$

If we now assume that the recurrence vector and skew symmetric part of the non-symmetric connection is also 1-recurrent with the same recurrence vector then from (4.5) we get

(4.6) $${}_{\ell m}\,{}^{+}|_{n}\, R^{i}_{jkh} = a_{n}\Big[R^{p}_{jkh} R^{i}_{p\ell m} - R^{i}_{pkh} R^{p}_{j\ell m} - R^{i}_{jph} R^{p}_{k\ell m} - R^{i}_{jkp} R^{p}_{h\ell m} - \\ -\left(\dot{\partial}_{p} R^{i}_{jkh}\right) R^{p}_{q\ell m}\dot{x}^{q} + a_{p} R^{i}_{jkh} N^{p}_{m\ell} + a_{p} R^{i}_{jkh} N^{p}_{m\ell}\Big].$$

we now use (3.13) in (4.6) and get

(4.7) ${}_{\ell m}\,{}^{+}|_{n} = a_{n} a_{p} N^{p}_{m\ell}.$

Now after making use of (4.3), (4.4) and (4.7), we get

(4.8) $\sim a_{r} N^{r}_{m\ell} + {}_{pm} v^{p}\,{}^{+}|_{\ell} + {}_{\ell p} v^{p}\,{}^{+}|_{m} = 0.$

From (4.2), we have

(4.9) $\sim{}^{+}|_{mn} = -\left({}_{ms} v^{s}\right){}^{+}|_{n}.$

Interchanging the indices m and n in (4.9) and subtracting the equation thus obtained from (4.9) itself, we get

(4.10) $\sim{}^{+}|_{mn} - \sim{}^{+}|_{nm} = \left({}_{ns} v^{s}\right){}^{+}|_{m} - \left({}_{ms} v^{s}\right){}^{+}|_{n}$

$\sim$ -being a non-constant scalar function, therefore under this assumption from (4.10), we get

(4.11) ${}_{sn}\,{}^{+}|_{m}\, v^{s} + {}_{ms}\,{}^{+}|_{n}\, v^{s} = {}_{ns} v^{s}\,{}^{+}|_{m} - {}_{ms} v^{s}\,{}^{+}|_{n},$

where we have written ${}_{ns} = - {}_{sn}$.

Using (4.11) into the left hand side of (4.8) and then again using (4.7) in the result thus obtained, we get

(4.12) $v^{p}\left(a_{m} N^{r}_{m\ell} + a_{\ell} N^{r}_{mp}\right) - \sim N^{r}_{m\ell} = 0.$

Thus, we can state

Theorem (4.1)

In a R^{+}-recurrent F_n^{*} the necessary and sufficient condition for the existence of (3.11) is given by (4.12).

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Conformal Motion in A R^+- Recurrent F_n^*

If the infinitesimal transformation (2.1) implies that the magnitude of the vectors defined in the same tangent space are proportional and the angle between the two directions is also the same with respect to the metrics then it will be called a conformal motion in F_n^*. The variation of the non-symmetric connection $\Gamma^i_{jk}(x,\dot{x})\left(\neq \Gamma^i_{kj}(x,\dot{x})\right)$ under the infinitesimal transformation (2.1) is $£_v \Gamma^i_{jk}$ and that under the conformal change is $\overline{\Gamma}^i_{jk}$. The two transformations will coincide if the corresponding variations are the same.

The necessary and sufficient condition in order that the infinitesimal point transformation (2.1) may define R^+-conformal motion is given by

(5.1) $$£_v \Gamma^i_{jk} = ¢^i_j \in_k + ¢^i_k \in_j - \in^i g_{jk},$$

Where $\in^i(x)$ is a non-zero contravariant vector field and it satisfies

(5.2) **(a)** $$\in^i = g^{ik} \in_k$$

(b) $$\in_k \dot{x}^k = 0$$

Now, we shall consider the conditions under which a R^+-conformal motion becomes R^+-curvature collineation. In view of (5.1), the commutation formula (2.7) gives

(5.3) $$\begin{aligned} £_v R^i_{hjk} &= ¢^i_h\left(\in_j{}^+|_k - \in_k{}^+|_j\right) + \in^i\left(g_{hk}{}^+|_j - g_{hj}{}^+|_k\right) + ¢^i_j \in_h{}^+|_k - \\ &\quad - ¢^i_k \in_h{}^+|_j - \in^{i+}|_k\, g_{hj} + \in^{i+}|_j\, g_{hk} - \left(\Gamma^i_{\ell hj} \in_k + \Gamma^i_{\ell hk} \in_j\right)\dot{x}^\ell + \\ &\quad + \dot{x}^\ell \in^r \left(\Gamma^i_{rhk} g_{ij} + \Gamma^i_{rhj} g_{ik}\right) N^r_{kj} \in_r ¢^i_h - N^i_{kj} \in_n + \in^i g_{hr} N^r_{kj}, \end{aligned}$$

where we have made use of (5.2b).

At this stage, we assume that the space is affinely connected then (5.3) with the help of (2.9) gives

(5.4) $$\begin{aligned} £_v R^i_{hjk} &= ¢^i_h\left(\in_j{}^+|_k - \in_k{}^+|_j\right) + \in^i\left(g_{hk}{}^+|_j - g_{hj}{}^+|_k\right) + \\ &\quad + ¢^i_j \in_h{}^+|_k - ¢^i_k \in_h{}^+|_j - \in^{i+}|_k\, g_{hj} + \in^{i+}|_j\, g_{hk} - \\ &\quad N^r_{kj} \in_r ¢^i_h - N^i_{kj} \in_h + \in^i g_{hr} N^r_{kj}. \end{aligned}$$

We now give the following definitions:

Definition (5.1)

The infinitesimal point transformation (2.1) defines a R^+-curvature collineation provided that the space F_n^* admits a vector field $v^i(x)$ such that

(5.5) $$£_v R^i_{jkh} = 0.$$

Definition (5.2)

The infinitesimal points transformation (2.1) defines a R^+-Ricci collineation provided there exists a vector field $v^i(x)$ such that

(5.6) $$£_v R_{jk} = 0 \text{ where } R_{jk} = R^i_{jki}.$$

We now make use of (5.5) in (5.4) and get

(5.7) $$¢^i_h\left(\in_j{}^+|_k-\in_k{}^+|_j\right)+\in^i\left(g_{hk}{}^+|_j-g_{hj}{}^+|_k\right)+¢^i_j\in_h{}^+|_k-$$
$$-¢^i_k\in_h{}^+|_j-\in^{i+}|_k\,g_{hj}+\in^{i+}|_j\,g_{hk}-N^r_{kj}\in_r¢^i_h-N^i_{kj}\in_h+$$
$$+\in^i g_{hr}N^r_{kj}=0.$$

We now contract (5.7) with respect to the indices i and h and get

(5.8) $$(n+1)\left(\in_j{}^+|_k-\in_k{}^+|_j\right)-(n+1)N^r_{kj}\in_r+$$
$$+\in^i\left(g_{ik}{}^+|_j-g_{ij}{}^+|_k\right)-\in^{i+}|_k\,g_{ij}+\in^{i+}|_j\,g_{ik}+$$
$$+\in^i g_{ir}N^r_{kj}=0.$$

Therefore, we can state

Theorem (5.1)

In an affinely connected $R^+-F^*_n$, in order that R^+-conformal motion be a R^+-curvature collineation (5.8) should necessarily hold.

We now contract the equation (5.3) with respect to the indices i and k and thereafter use (5.6) and get

(5.9) $$£_v R_{hj}=\in_j{}^+|_h-n\in_h{}^+|_j+\in^i\left(g_{hi}{}^+|_j-g_{hj}{}^+|_i\right)-$$
$$-\in^{i+}|_i\,g_{hj}+\in^{i+}|_j\,g_{hi}-A_{hj}-\Gamma^i_{rhi}\in_j\dot{x}^\ell+$$
$$+\left(B_{hj}+C_{hj}-D_{hj}-E_{jh}+F_{hj}\right)$$

where

(5.10) **(a)** $A_{hj}=\Gamma^i_{\ell hj}\in_i\dot{x}^\ell$,

(b) $B_{hj}=\Gamma^i_{rhi}g_{\ell j}\in^r\dot{x}^\ell$,

(c) $C_{hj}=\Gamma^i_{rhj}g_{\ell i}\in^r\dot{x}^\ell$,

(d) $D_{hj}=N^r_{hj}\in_r$,

(e) $E_{jh}=N^i_{ij}\in_h$,

(f) $F_{hj}=\in^i g_{hr}N^r_{ij}$.

We now use (5.6) in (5.9) and get

(5.11) $$\in_j{}^+|_h-n\in_h{}^+|_j+\in^i\left(g_{hi}{}^+|_j-g_{hj}{}^+|_i\right)-\in^{i+}|_i\,g_{hj}+\in^{i+}|_j\,g_{hi}-$$
$$-\left(A_{hj}-B_{hj}-C_{hj}-F_{hj}+D_{hj}+E_{jh}\right)-\Gamma^i_{rhi}\in_j\dot{x}^r=0.$$

At this stage, we now assume that the space under consideration is affinely connected, therefore under this assumption, we get

(5.12) $$C_{hj}=A_{hj}=B_{hj}=0$$

using (5.12) in (5.11), we get

(5.13) $$\in_j{}^+|_h-n\in_{h+}{}^+|_j+\in^i\left(g_{hi}{}^+|_j-g_{hj}{}^+|_i\right)-\in^{i+}|_i\,g_{hj}+\in^{i+}|_j\,g_{hi}-$$
$$-\left(D_{hj}+E_{jh}-F_{hj}\right)-\Gamma^i_{rhi}\in_j\dot{x}^r=0.$$

Hence, we can state:

Theorem (5.2)

In an affinely connected $R^{+}-F_{n}^{*}$, in order that the R^{+}-conformal motion be Ricci collineation, the equation (5.13) must necessarily hold.

Special Projective Motion In A $R^{+}-F_{n}^{*}$

If the infinitesimal point transformation (2.1) leaves invariant the system of geodesics into the same system then it is called a R^{+}-projective transformation and the transformation (2.1) is said to define a R^{+}-special projective motion if the Lie-derivative of the non-symmetric connection coefficient $\Gamma^{i}_{jk}(x,\dot{x})$ satisfies

(6.1) $\text{£}_{v}\Gamma^{i}_{jk}=\text{¢}^{k}_{j}=\text{¢}^{i}_{k}+\in^{i}\text{a}_{jk}$,

where a is an arbitrarily chosen positively homogeneous scalar function of degree one in its directional arguments and satisfies the following relations:

(6.2) **(a)** $\text{a}_{k}=\dot{\partial}_{k}\text{a}$

(b) $\text{a}_{jk}=\dot{\partial}_{j}\dot{\partial}_{k}\text{a}$

(c) $\text{a}_{k}\dot{x}^{k}=\text{a}$

(d) $\text{a}_{jk}\dot{x}^{k}=0$.

using (6.1) into the commutation formula (2.7), we get

(6.3)
$$\begin{aligned}\text{£}_{v}R^{i}_{hjk}=&\text{¢}^{i}_{h}\text{a}_{j}{}^{+}|_{k}+\text{¢}^{i}_{j}\text{a}_{h}{}^{+}|_{k}-\text{¢}^{i}_{h}\text{a}_{k}{}^{+}|_{j}-\text{¢}^{i}_{k}\text{a}_{h}{}^{+}|_{j}+\text{a}_{hj}{}^{+}|_{k}\dot{x}^{i}+\\&+\text{a}_{hk}{}^{+}|_{j}\dot{x}^{i}-\Gamma^{i}_{rhj}\text{a}_{k}\dot{x}^{r}-\text{a}\Gamma^{i}_{khj}+\Gamma^{i}_{rhk}\text{a}_{j}\dot{x}^{r}+\text{a}\Gamma^{i}_{jhk}-\\&-N^{r}_{kj}\text{¢}^{i}_{h}\text{a}_{r}-N^{i}_{kj}\text{a}_{h}-N^{r}_{kj}\text{a}_{hr}\dot{x}^{i},\end{aligned}$$

where, we have taken into account (6.2) and the fact that the covariant derivative of ¢^{i}_{h} and $\dot{x}^{i}$ vanish identically. At this stage, we assume that the space F_{n}^{*} admits R^{+}-curvature collineation, then using (5.5) in (6.3), we get

(6.4)
$$\begin{aligned}&\text{¢}^{i}_{h}\text{a}_{j}{}^{+}|_{k}-\text{¢}^{i}_{j}\text{a}_{h}{}^{+}|_{k}+\dot{x}^{i}\text{a}_{hj}{}^{+}|_{k}-\text{¢}^{i}_{h}\text{a}_{k}{}^{+}|_{j}-\text{¢}^{i}_{k}\text{a}_{h}{}^{+}|_{j}+\\&+\dot{x}^{i}\text{a}_{hj}{}^{+}|_{k}-\Gamma^{i}_{rhj}\dot{x}^{r}\text{a}_{k}-\text{a}\Gamma^{i}_{khj}+\Gamma^{i}_{rhk}\dot{x}^{r}\text{a}_{j}+\text{a}\Gamma^{i}_{jhk}-\\&-N^{r}_{kj}\text{¢}^{i}_{h}\text{a}_{r}-N^{i}_{kj}\text{a}_{h}-N^{r}_{kj}\text{a}_{hr}\dot{x}^{r}=0.\end{aligned}$$

Contracting (6.4) with respect to the indices i and j, we get

(6.5)
$$\begin{aligned}&n\,\text{a}_{h}{}^{+}|_{k}-\text{a}_{k}{}^{+}|_{h}+\text{a}_{hk}{}^{+}|_{i}\dot{x}^{i}-\text{a}\left(\Gamma^{i}_{khi}-\Gamma^{i}_{ihk}\right)-\\&-\dot{x}^{r}\left(\Gamma^{i}_{rhi}\text{a}_{r}-\Gamma^{i}_{rhk}\text{a}_{r}\right)-N^{r}_{kh}\text{a}_{r}-N^{r}_{ki}\text{a}_{h}-N^{r}_{ki}\text{a}_{hr}\dot{x}^{i}=0.\end{aligned}$$

Therefore, we can state:

Theorem (6.1)

In a $R^{+}-F_{n}^{*}$, in order that R^{+}- special projective motion be R^{+}- curvature collineation (6.5) must necessarily hold.

We now contract, (6.3) with respect to the indices i and k, we get

(6.6) $$\pounds_{v} R_{hj} = a_{j}{}^{+}|_{h} - a_{h}{}^{+}|_{j} - \Gamma^{i}_{rhj} a_{i} \dot{x}^{r} - a\left(\Gamma^{i}_{hji} - \Gamma^{i}_{jhi}\right) + \Gamma^{i}_{rhi} \dot{x}^{r} a_{j} - N^{r}_{hj} a_{r} - N^{i}_{ij} a_{h} - N^{r}_{ij} a_{hr} \dot{x}^{i}.$$

At this stage, we suppose that the infinitesimal transformation (2.1) defines a R^{+}-Ricci collineation characterised by (5.6), then from (6.6), we get

(6.7) $$a_{j}{}^{+}|_{h} - n\, a_{h}{}^{+}|_{j} - \Gamma^{i}_{rhj} a_{i} \dot{x}^{r} - a\left(\Gamma^{i}_{hji} - \Gamma^{i}_{jhi}\right) + \Gamma^{i}_{rhi} \dot{x}^{r} a_{j} - N^{r}_{hj} a_{r} - N^{i}_{ij} a_{h} - N^{r}_{ij} a_{hr} \dot{x}^{i} = 0$$

Therefore, we can state

Theorem (6.2)

In a $R^{+} - F^{*}_{n}$, in order that R^{+}- special projective motion be R^{+}-Ricci collineation (6.7) must necessarily hold.

At the stage, if we assume that the space under consideration is affinely connected then under this assumption (6.5) and (6.7) assume the following alternative forms

(6.8) $$n\, a_{h}{}^{+}|_{k} - a_{k}{}^{+}|_{h} + a_{hk}{}^{+}|_{i}\, \dot{x}^{i} - N^{r}_{kh} a_{r} - N^{r}_{ki} a_{h} - N^{r}_{ki} a_{hr} \dot{x}^{i} = 0,$$

And

(6.9) $$a_{j}{}^{+}|_{h} - n\, a_{h}{}^{+}|_{j} - N^{r}_{hj} a_{r} - N^{i}_{ij} a_{h} - N^{r}_{ij} a_{hr} \dot{x}^{i} = 0.$$

Thus, we can state:

Theorem (6.3)

In an affinely connected $R^{+} - F^{*}_{n}$ in order that R^{+}-special projective motion be R^{+}-curvature collineation (6.8) must necessarily hold.

Theorem (6.4)

In an affinely connected $R^{+} - F^{*}_{n}$, in order that R^{+}-special projective motion be R^{+}-Ricci collineation (6.9) must necessarily hold.

References

1. Cartan, E.: Les espaces de Finsler,Actualities 79, Paris 1934, 41p, 2nd edit., 1971.
2. Gupta, K. K.: Certain investigations in the differential geometry of Finsler spaces equipped with non – symmetric connection Ph.D. Thesis, Univ. of Gorakhpur, India (1980).
3. Kumar, A.: On the existence of projective affine motion in a projective recurrent Finsler space, Acta Ciencia Indica, vol. 3(3), 243-246 (1977).
4. Pande, H. D& Kumar, and A.: Special infinitesimal projective transformation and curvature collimation in Finsler space, Acad. Naz. De. Lin. Ser. VIII, vol.57, 1173 - 1179 (1971).
5. Rund, H.:The differential geometry of Finsler spaces, Springer Verlag (Berlin), 1959.
6. Singh, U. P & Singh, A.K.: On the N-curvature collineationin a Finsler space, Ann.de la Soc. Scientifique, de Bruxelles T. 95, 69-77 (1981).
7. Takano, K.: On the existence of affine motion in a space with recurrent curvature, Tensor (N.S) (1) 17, 68-73 (1966).
8. Vranceanu, Gh.: Lectii dee geometric differentia la, Vol I, EDP, BUC., 1962.
9. Yano, K.: The theory of Lie-derivatives and its applications, North Holland Publ. Co., Amesterdam (1957).